Topics in complex analysis. Lecture 4 (Oct 7, 2014)

Proof of Lemma 3.2. wts

1. $\exists \prod_{j=m}^{\infty} a_j =: a(m) \; \forall m$

2. $a(m) \longrightarrow 1$ as $m \to \infty$

3. $a_j \longrightarrow 1$ as $j \to \infty$

wlog $a_j \neq 0 \; \forall j$. (acc to Def. 3.1). The existence of $a(j_0)$ where j_0 is from Def. 3.1, follows from the def. Multiplying or dividing finitely many a_j's yields 1.

For 2. we notice

$$\frac{a}{a(m)} = \lim_{n \to \infty} \frac{\prod_{j=1}^{n} a_j}{\prod_{j=m}^{n} a_j} = \cancel{\lim_{n\to\infty}} \underbrace{\prod_{j=1}^{m-1} a_j}$$

Letting $m \to \infty$ here: $\longrightarrow a$

Hence by algebraic manipulations

$$\lim_{m \to \infty} a(m)\, a = a \xrightarrow{a \neq 0} \lim_{m \to \infty} a(m) = 1.$$

3. follows by observing $a_j = \dfrac{a(j)}{a(j+1)} \; \forall j$.

Take limits in this equality. □

Recap: complex logarithm.

Problem For $z \in \mathbb{C} \setminus \{0\}$ the eqn $e^{\omega} = z$ has infinitely many solutions (since $e^{2\pi i k} = 1$, $k \in \mathbb{Z}$)

Rem write $z = re^{i\varphi}$. Set $\omega_n = \log r + i(\varphi + 2\pi n)$

Then $e^{\omega_n} = z$ and

$$\exp^{-1}(z) = \{\omega_n : n \in \mathbb{N}\}.$$

Rem Since $(e^z)' = e^z \neq 0$ the equ $e^w = z$ is uniquely solvable at least locally (by inverse fct thm)

Def (Principal branch of log) $\mathbb{C}^- := \mathbb{C} \setminus (-\infty, 0]$.
For $z \in \mathbb{C}^-$ we write $z = re^{i\varphi}$, $-\pi < \varphi < \pi$.

$$\log(z) = \log r + i\varphi .$$

Proof of Lemma 3.3. "$\Leftarrow$" Assume $\sum_j \log(a_j)$ exists.
Apply exp to this sum:

$$0 \neq \boxed{\exp\left(\sum_{j=1}^{\infty} \log a_j\right)} = \exp\left(\lim_{n\to\infty} \sum_{j=1}^{n} \log a_j\right)$$

$$\underset{\text{cts}}{\overset{\exp}{\hookrightarrow}}= \lim_{n\to\infty} \exp\left(\sum_{j=1}^{n} \log a_j\right) = \boxed{\lim_{n\to\infty} \prod_{j=1}^{n} a_j}$$

j_0

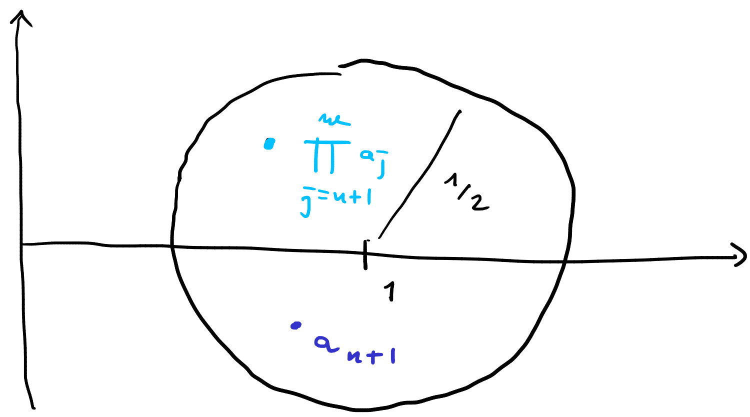
"$\Rightarrow$" Assume $\prod_{j=1}^{\infty} a_j$ exists.

Since it is not equal to zero $\exists n_0 \in \mathbb{N}$ $\forall m \geq n \geq n_0$:

$$\underbrace{|P_n - P_m|}_{\text{conv to zero}} \leq \frac{1}{2} \underbrace{|P_n|}_{\text{converges, but not to zero}} \qquad \left(P_n = \prod_{j=1}^{n} a_j .\right)$$

equiv.

$$\left|1 - \underbrace{\frac{P_m}{P_n}}_{\prod_{j=n+1}^{m} a_j}\right| \leq \frac{1}{2}$$



Since in particular, all a_j's and their product, $j \ge n_0$, are contained in a region where the logarithm rules hold for the principal branch of log, we have

$$\log\left(\prod_{j=n_0+1}^{m} a_j\right) = \sum_{j=n_0+1}^{m} \log a_j$$

Let $m \to \infty$ and use cty of log. $\square$

Proof of Corollary 3.7. (i) WTS

1. $\prod_{j=n}^{\infty} f_j \longrightarrow 1$ as $n \to \infty$ loc unif cvg
2. $f_j \longrightarrow 1$ as $j \to \infty$

By Lemma 3.2 the number $g_n(z) = \prod_{j=n}^{\infty} f_j(z)$ is well-defined $\forall z \in U$. Def. a fct $g_n : U \to \mathbb{C}$

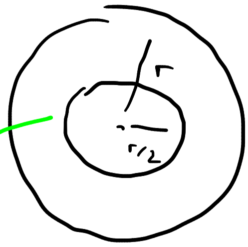
Let $z_0 \in U$ and $r > 0$ and $j_0 \in \mathbb{N}$ as in Def. 3.6

g_{j_0} is the locally unif limit of $\Big(\underbrace{\prod_{j=n}^{m} f_j}_{\text{cty!}}\Big)_{m \in \mathbb{N}}$

it is continuous.

Observe $g_{j_0}(z) \neq 0 \quad \forall z \in B_r(z_0)$ (by definition 3.6)

In particular $\underbrace{\inf_{z \in B_{r/2}(z_0)} |g_{j_0}(z)| = 2c > 0}$



Since $\prod_{j=j_0}^{n-1} f_j \longrightarrow g_{j_0}$ unif on $B_r(z_0)$

$\exists n_0 \in \mathbb{N} \ \forall n \ge n_0 \quad \underbrace{\inf_{z \in B_{r/2}(z_0)} \left| \prod_{j=j_0}^{n-1} f_j(z) \right| \ge c}_{(*)}$

Then for $n \geq \max\{j_0, n_0\}$ and $z \in B_{r/2}(z_0)$

$$|g_n(z) - 1| = \left| \frac{g_{j_0}(z)}{\prod_{j=j_0}^{n-1} f_j(z)} - 1 \right|$$

$$= \underbrace{\frac{1}{\left|\prod_{j=j_0}^{n-1} f_j(z)\right|}}_{\leq \frac{1}{c} \text{ by } (*)} \left| g_{j_0}(z) - \prod_{j=j_0}^{n-1} f_j(z) \right|$$

$$\leq \frac{1}{c} \left| g_{j_0}(z) - \prod_{j=j_0}^{n-1} f_j(z) \right|$$

$$\leq \frac{1}{c} \underbrace{\sup_{z \in B_{r/2}(z_0)} \left| g_{j_0}(z) - \prod_{j=j_0}^{n-1} f_j(z) \right|}_{\to 0 \text{ by loc unif cvg.}}$$

Taking the supremum over $z \in B_{r/2}(z_0)$ on the lhs and letting $n \to \infty$ gives $g_n \to 1$ unif on $B_{r/2}(z_0)$ (hence $g_n \to 1$ loc unif by arbitrariness of z_0).

The loc unif cvg of f_j to 1 as $j \to \infty$ follows from the previous part and the identity $f_j = \dfrac{g_j}{g_{j+1}}$.

(ii) Locally unif conv. and (i) implies the following. For every cpt subset $K \subseteq U$ $\exists j^* \in \mathbb{N}$

$$\sup_{z \in K} \left| \prod_{j=j^*}^{n} f_j - f \right| \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\sup_{z \in K} \left| \prod_{j=j^*}^{n} g_j - g \right| \longrightarrow 0 \quad \text{as } n \to \infty$$

This implies

$$\sup_{z \in K} \left| \prod_{j=j^*}^{n} f_j g_j - fg \right|$$

$$= \sup_{z \in K} \left| \left(\prod_{j=j^*}^{n} f_j \right) \left(\prod_{j=j^*}^{n} g_j \right) - fg \right|$$

$$\leq \sup_{z \in K} \left| \left(\prod_{j=j^*}^{n} f_j \right) \left(\prod_{j=j^*}^{n} g_j \right) - \underline{\left(\prod_{j=j^*}^{n} f_j \right) g} \right|$$

$$+ \sup_{z \in K} \left| \underline{\left(\prod_{j=j^*}^{n} f_j \right) g} - fg \right|$$

$$\leq C \underbrace{\sup_{z \in K} \left| \prod_{j=j^*}^{n} g_j - g \right|}_{\to 0 \text{ as } n \to \infty} + C \underbrace{\sup_{z \in K} \left| \prod_{j=j^*}^{n} f_j - f \right|}_{\to 0 \text{ as } n \to \infty}$$

Since $fg \neq 0$, we obtain the desired cvg.

(iii) Thm 1.3. □

Proof of Lemma 3.9

Let $z_0 \in U$. wts. $\exists r > 0$ and $\exists j_0 \in \mathbb{N}$ st.

$$\prod_{j=j_0}^{\infty} (1 + g_j) \text{ conv. unif of } B_r(z_0)$$

Since log of $\sum_j g_j$ loc normally convergent $\exists r > 0$ st

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(z_0)} |g_j(z)| < \infty$$

Since summable sequ conv to zero, $\exists j_0 \in \mathbb{N}\ \forall j \geq j_0$:

$$\sup_{z \in B_r(z_0)} |g_j(z)| \leq \delta$$

where $\delta > 0$ is chosen st $\frac{1}{2}|z| \leq |\log(1+z)| \leq 2|z|$
$\forall z \in B_\delta(0)$

By these preparations.

$$\sum_{j=j_0}^{\infty} \sup_{z \in B_r(z_0)} |\log(1+g_j(z))| \leq 2 \sum_{j=j_0}^{\infty} \sup_{z \in B_r(z_0)} |g_j(z)| < \infty$$

Lemma 1.11 $\Longrightarrow$

$$\sum_{j \geq j_0} \log(1+g_j) \text{ conv loc unif.}$$

Taking exp yields the claim. $\square$